Exercise E5.10

$$y[n+2] - \frac{5}{6}y[n+1] + \frac{1}{6}y[n] = 5x[n+1] - x[n]$$

$x[n] = u[n]$

$y[-1] = 2$

$y[-2] = 0$

Zero input

$$y_0[n+2] - \frac{5}{6}y_0[n+1] + \frac{1}{6}y_0[n] = 0$$

$$\left(z^2 - \frac{5}{6}z + \frac{1}{6}\right) y_0[n] = 0$$

$$\left(z-\tfrac{1}{2}\right)\left(z-\tfrac{1}{3}\right) = 0 \rightarrow z = \left\{\tfrac{1}{2}, \tfrac{1}{3}\right\} \rightarrow y_0[n] = a\,2^{-n} + b\,3^{-n}$$

$$y_0[-1] = 2 = a\,2 + b\,3$$

$$y_0[-2] = 0 = a\,2^2 + b\,3^2 \rightarrow \begin{bmatrix} 2 & 3 & | & 2 \\ 4 & 9 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & | & 3 \\ 0 & 1 & | & -4/3 \end{bmatrix} \rightarrow \begin{matrix} a = 3 \\ b = -4/3 \end{matrix}$$

$$y_0[n] = 3\,2^{-n} - \frac{4}{3}3^{-n}$$

Zero state

$$\left(z^2 - \frac{5}{6}z + \frac{1}{6}\right) Y_p(z) = (5z-1)X(z)$$

$$Y_p(z) = \frac{5z-1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)} X(z) = H(z)X(z)$$

$$X(z) = \mathcal{Z}\{u[n]\} = \frac{z}{z-1}$$

degree (numerator) < degree (denominator)

$$\frac{Y_p(z)}{z} = \frac{5z-1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)(z-1)} = \frac{A}{z-\frac{1}{2}} + \frac{B}{z-\frac{1}{3}} + \frac{C}{z-1}$$

$$A = \lim_{z\to\frac{1}{2}} \frac{Y_p(z)}{z}\left(z-\tfrac{1}{2}\right) = \lim_{z\to\frac{1}{2}} \frac{5z-1}{\left(z-\frac{1}{3}\right)(z-1)} = \frac{5/2 - 2/2}{\left(\frac{1}{6}\right)\left(-\frac{1}{2}\right)} = -\frac{3\cdot 12}{2} = -18$$

$$B = \lim_{z\to\frac{1}{3}} \frac{Y_p(z)}{z}\left(z-\tfrac{1}{3}\right) = \lim_{z\to\frac{1}{3}} \frac{5z-1}{\left(z-\frac{1}{2}\right)(z-1)} = \frac{5/3 - 3/3}{\left(-\frac{1}{6}\right)\left(-\frac{2}{3}\right)} = \frac{(6)(2)}{2} = 6$$

$$C = \lim_{z\to 1} \frac{Y_p(z)}{z}(z-1) = \lim_{z\to 1} \frac{5z-1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)} = \frac{5-1}{\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)} = 4\cdot 3 = 12$$

$$Y_p(z) = -18\frac{z}{z-\frac{1}{2}} + 6\frac{z}{z-\frac{1}{3}} + 12\frac{z}{z-1} \xrightarrow{\mathcal{Z}^{-1}} y_p[n] = \left(-18\cdot 2^{-n} + 6\cdot 3^{-n} + 12\right)u[n]$$

$$y[n] = y_0[n] + y_p[n] = \left(-18+3\right)2^{-n} + \left(6 - \frac{4}{3}\right)3^{-n} + 12\Big)u[n] = \boxed{\left(-15\cdot 2^{-n} + \frac{14}{3}\cdot 3^{-n} + 12\right)u[n]}$$